Exercises

5.1.1 Given that $\triangle ABC \sim \triangle YXZ$, which of the statements below must be true?

(a) $AB/YX = AC/YZ$.

(b) $AB/BC = YX/XZ$.

(c) $AB/XZ = BC/YX$.

(d) $(AC)(YX) = (YZ)(BA)$.

(e) $BC/BA = XY/ZY$.

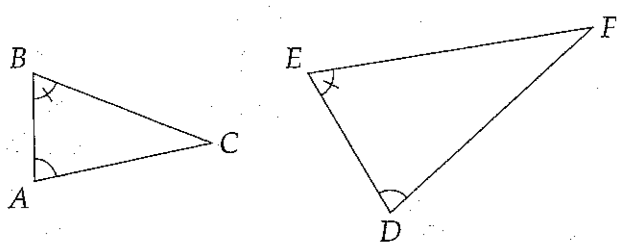
5.1.2 $\triangle ABC \sim \triangle ADB$, $AC = 4$, and $AD = 9$. What is AB? *(Source: MATHCOUNTS)* **Hints:** 113

5.2 AA Similarity

In our introduction, we stated that similar figures have all corresponding angles equal, and that corresponding sides are in a constant ratio. It sounds like a lot of work to prove all of that; however, just as for triangle congruence, we have some shortcuts to prove that triangles are similar. We'll start with the most commonly used method.

Important: **Angle-Angle Similarity (AA Similarity)** tells us that if two angles of one triangle equal two angles of another, then the triangles are similar. $\angle A = \angle D$ and $\angle B = \angle E$ together imply $\triangle ABC \sim \triangle DEF$, so

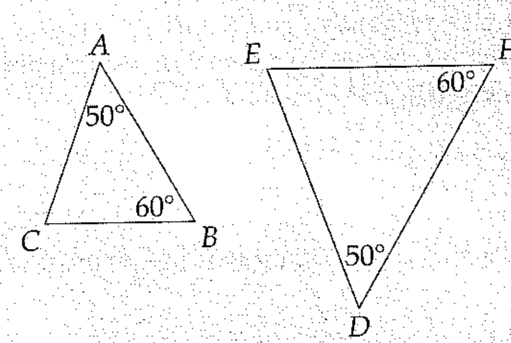
$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$



We'll explore why AA Similarity works in Section 5.5, but first we'll get some experience using it in some problems.

Problems

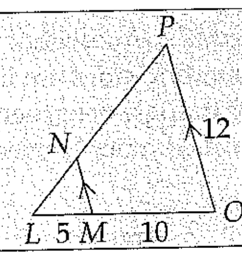
Problem 5.2: Below are two triangles that have the same measures for two angles.



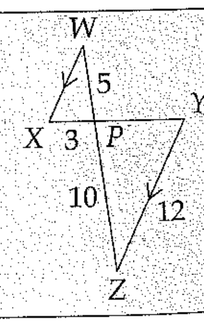
Find the third angle in each, and find the ratios AB/DF, AC/DE, BC/EF by measuring the sides with a ruler.

Problem 5.3: In this problem we try to extend AA Similarity to figures with more angles by considering figures with four angles. Can you create a figure $EFGH$ that has the same angles as $ABCD$ at right such that $EFGH$ and $ABCD$ are not similar? (In other words, can you create $EFGH$ so that the angles of $EFGH$ equal those of $ABCD$, but the ratio of corresponding sides between $EFGH$ and $ABCD$ is not the same for all corresponding pairs of sides?)

Problem 5.4: In the figure at right, $\overline{MN} \parallel \overline{OP}$, $OP = 12$, $MO = 10$, and $LM = 5$. Find MN.



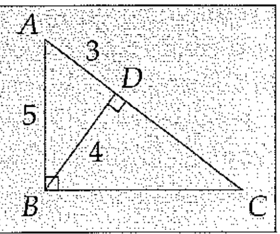
Problem 5.5: The lengths in the diagram are as marked, and $\overline{WX} \parallel \overline{YZ}$. Find PY and WX.



Extra! My dad was going to cut down a dead tree in our yard one day, but he was afraid it might hit some nearby power lines. He knew that if the tree were over 45 feet tall, the tree would hit the power lines. He stood 30 feet from the base of the tree and held a ruler 6 inches in front of his eye.

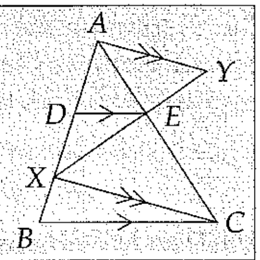
Continued on the next page. . .

Problem 5.6: Find BC and DC given $AD = 3$, $BD = 4$, and $AB = 5$.

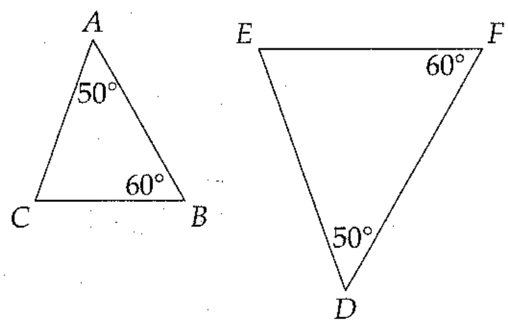


Problem 5.7: Given that $\overline{DE} \parallel \overline{BC}$ and $\overline{AY} \parallel \overline{XC}$, prove that

$$\frac{EY}{EX} = \frac{AD}{DB}.$$



Problem 5.2: Below are two triangles that have the same measures for two angles.



Find the third angle in each, and find the ratios AB/DF, AC/DE, BC/EF by measuring the sides with a ruler.

Solution for Problem 5.2: The last angle in each triangle is $180° - 50° - 60° = 70°$, so the angles of $\triangle ABC$ match those of $\triangle DFE$. In the same way, if we ever have two angles of one triangle equal to two angles of another, we know that the third angles in the two triangles are equal.

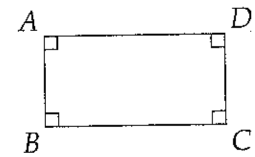
Measuring, we find that the ratios are each 1.5. It appears to be the case that if all the angles of two triangles are equal, then the two triangles are similar. □

We might wonder if two figures with equal corresponding angles are always similar. So, we add an angle and see if it works for figures with four angles.

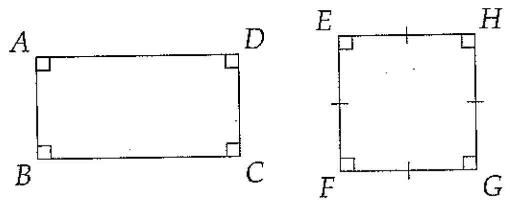
Extra! *. . . continued from the previous page*

He lined the bottom of the ruler up with the base of the tree, and saw that the top of the tree lined up with a point 8 inches high on the ruler. He then knew he could safely cut the tree down. How did he know?

Problem 5.3: Does your rule work for figures with more than 3 angles? Can you create a figure $EFGH$ that has the same angles as $ABCD$ at right such that $EFGH$ and $ABCD$ are not similar? (In other words, can you create $EFGH$ so that the angles of $EFGH$ equal those of $ABCD$, but the ratio of corresponding sides between $EFGH$ and $ABCD$ is not the same?)



Solution for Problem 5.3: We can quickly find such an $EFGH$. The diagram to the right shows a square $EFGH$ next to our initial rectangle. Clearly these figures have the same angles, but when we check the ratios, we find that

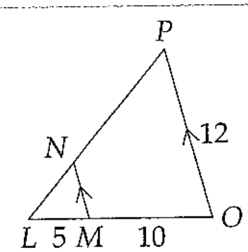


$$\frac{AB}{EF} < 1 < \frac{BC}{FG}.$$

$ABCD$ and $EFGH$ are not similar, so equal angles are not enough to prove similarity here. □

Let's return to triangles and tackle some problems using AA Similarity.

Problem 5.4: In the figure at right, $\overline{MN} \parallel \overline{OP}$, $OP = 12$, $MO = 10$, and $LM = 5$. Find MN.

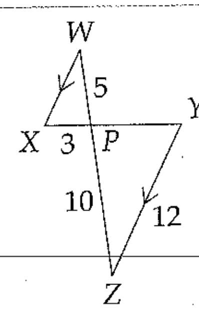


Solution for Problem 5.4: See if you can find the flaw in this solution:

Bogus Solution: Since $\overline{MN} \parallel \overline{OP}$, we have $\angle LMN = \angle LOP$ and $\angle LNM = \angle LPO$. Therefore, $\triangle LMN \sim \triangle LOP$, so $LM/MO = MN/OP$. Substituting our given side lengths gives $5/10 = MN/12$, so $MN = 6$.

Everything in this solution is correct except for $LM/MO = MN/OP$. $\overline{MO}$ is not a side of one of our similar triangles! The correct equation is $LM/LO = MN/OP$. Since $LO = LM + MO = 15$, we now have $5/15 = MN/12$, so $MN = 4$. □

Problem 5.5: The lengths in the diagram are as marked, and $\overline{WX} \parallel \overline{YZ}$. Find PY and WX.



Solution for Problem 5.5: Where does this solution go wrong:

Bogus Solution: Since $\overline{WX} \parallel \overline{ZY}$, we have $\angle W = \angle Z$ and $\angle X = \angle Y$. Therefore, $\triangle WPX \sim \triangle YPZ$, and we have

$$\frac{PX}{PZ} = \frac{WX}{YZ} = \frac{WP}{PY}$$

Substitution gives

$$\frac{3}{10} = \frac{WX}{12} = \frac{5}{PY}.$$

We can now easily find $YP = 50/3$ and $WX = 18/5$.

This solution doesn't get the vertex order in the similar triangles right, so it sets up the ratios wrong! $\overline{PX}$ and $\overline{PZ}$ are not corresponding sides. $\overline{PX}$ in $\triangle WPX$ corresponds to $\overline{PY}$ in $\triangle ZPY$ because $\angle W = \angle Z$.

Here's what the solution should look like. Pay close attention to the vertex order in the similarity relationship.

Since $\overline{WX} \parallel \overline{ZY}$, we have $\angle W = \angle Z$ and $\angle X = \angle Y$. Therefore, $\triangle WPX \sim \triangle ZPY$. Hence, we have

$$\frac{PX}{PY} = \frac{WX}{YZ} = \frac{WP}{PZ}.$$

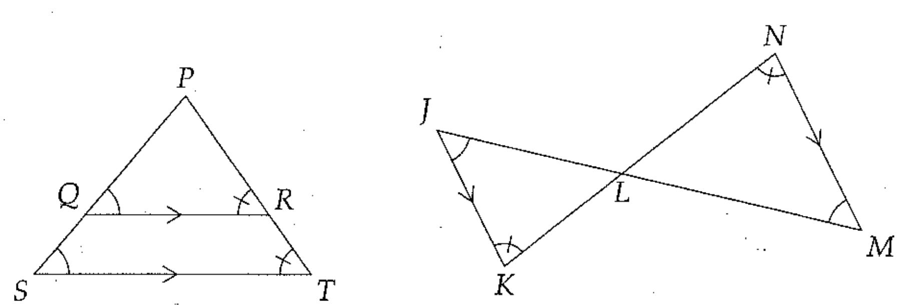
Substitution gives

$$\frac{3}{PY} = \frac{WX}{12} = \frac{5}{10}.$$

We can now easily find $PY = 6$ and $WX = 6$. □

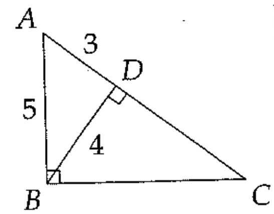
Perhaps you see a common thread in the last two problems. While you won't always find parallel lines in similar triangle problems, you'll almost always find similar triangles when you have parallel lines.

Important: Parallel lines mean equal angles. Equal angles mean similar triangles. The figures below show two very common set-ups in which parallel lines lead to similar triangles. Specifically, $\triangle PQR \sim \triangle PST$ and $\triangle JKL \sim \triangle MNL$.



WARNING!! Read the Bogus Solutions to Problems 5.4 and 5.5 again. These are very common errors; understand them so you can avoid them.

Problem 5.6: Find BC and DC given $AD = 3$, $BD = 4$, and $AB = 5$.



Solution for Problem 5.6: Since $\angle BAD = \angle CAB$ and $\angle BDA = \angle CBA$, we have $\triangle BAD \sim \triangle CAB$ by AA Similarity. Therefore, we have $BC/BD = AB/AD = 5/3$, so $BC = (5/3)(BD) = 20/3$.

We can use this same similarity to find AC, and then subtract AD to get CD. We could also note that $\angle BCD = \angle BCA$ and $\angle BDC = \angle CBA$, so $\triangle BCD \sim \triangle ACB$ by AA Similarity. Therefore, $CD/BD = BC/AB = (20/3)/5 = 4/3$, so $CD = (4/3)(BD) = 16/3$. □

Similar triangles – they're not just for parallel lines.

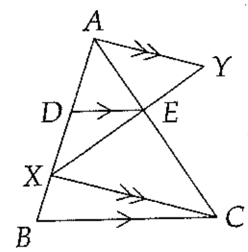
Important: Similar triangles frequently pop up in problems with right angles. The diagram in Problem 5.6 shows a common way this occurs. Make sure you see that

$$\triangle ABD \sim \triangle BCD \sim \triangle ACB.$$

As you'll see throughout the rest of the book, similar triangles occur in all sorts of problems, not just those with parallel lines and perpendicular lines. They're also an important step in many proofs.

Problem 5.7: Given that $\overline{DE} \parallel \overline{BC}$ and $\overline{AY} \parallel \overline{XC}$, prove that

$$\frac{EY}{EX} = \frac{AD}{DB}.$$



Solution for Problem 5.7: Parallel lines mean similar triangles. The ratios of side lengths in the problem also suggest we look for similar triangles.

Since $\overline{AY} \parallel \overline{XC}$, we have $\triangle AYE \sim \triangle CXE$. Now we look at what this means for our ratios. From $\triangle AYE \sim \triangle CXE$, we have $EY/EX = AE/EC$. All we have left is to show that $AE/EC = AD/DB$.

Since $\overline{DE} \parallel \overline{BC}$, we have $\triangle ADE \sim \triangle ABC$. Therefore, $AD/AB = AE/AC$, which is almost what we want! We break AB and AC into $AD + DB$ and $AE + EC$, hoping we can do a little algebra to finish:

$$\frac{AD}{AD + DB} = \frac{AE}{AE + EC}$$

If only we could get rid of the AD and AE in the denominators – then we would have $AD/DB = AE/EC$. Fortunately, we can do it. We can flip both fractions:

$$\frac{AD + DB}{AD} = \frac{AE + EC}{AE}.$$

Therefore, $\frac{AD}{AD} + \frac{DB}{AD} = \frac{AE}{AE} + \frac{EC}{AE}$, so $1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$, which gives us

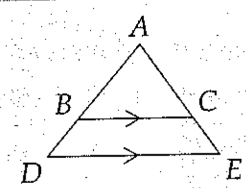
$$\frac{DB}{AD} = \frac{EC}{AE}.$$

Flipping these fractions back over gives us $AD/DB = AE/EC$. Therefore, we have $EY/EX = AE/EC = AD/DB$, as desired. □

Our solution to the previous problem reveals another handy relationship involving similar triangles:

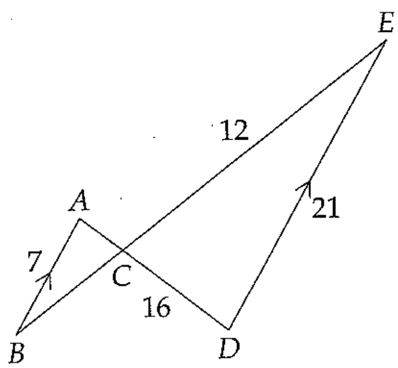
Important: If $\overline{BC} \parallel \overline{DE}$ and $\overleftrightarrow{BD}$ and $\overleftrightarrow{CE}$ meet at A as shown, then

$$\frac{AB}{BD} = \frac{AC}{CE}.$$

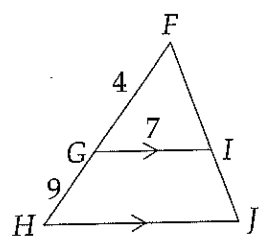


Exercises

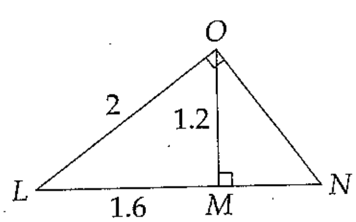
5.2.1



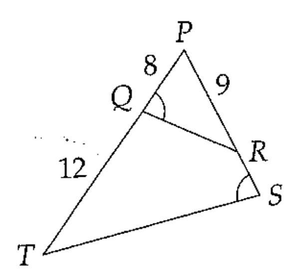
(a) Find AC and BC.



(b) Find HJ.

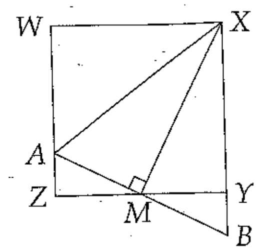


(c) Find ON and MN.



(d) Find RS.

5.2.2 If two isosceles triangles have vertex angles that have the same measure, are the two triangles similar? Why or why not?



5.2.3 In the diagram, $WXYZ$ is a square. M is the midpoint of $\overline{YZ}$, and $\overline{AB} \perp \overline{MX}$.

(a) Show that $\overline{WZ} \parallel \overline{XY}$. **Hints:** 182

(b) Prove that $AZ = YB$.

(c) Prove that $XB = XA$.

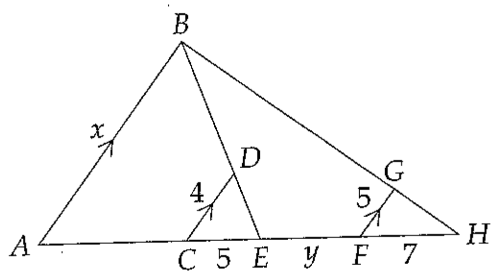
(d) Prove that $\triangle AZM \sim \triangle MYX$, and use this fact to prove $AZ = XY/4$.

5.2.4 In triangle ABC, $AB = AC$, $BC = 1$, and $\angle BAC = 36^\circ$. Let D be the point on side $\overline{AC}$ such that $\angle ABD = \angle CBD$.

(a) Prove that triangles ABC and BCD are similar.

(b)★ Find AB. **Hints:** 150

5.2.5★ Find x in terms of y given the diagram below. **Hints:** 258, 522



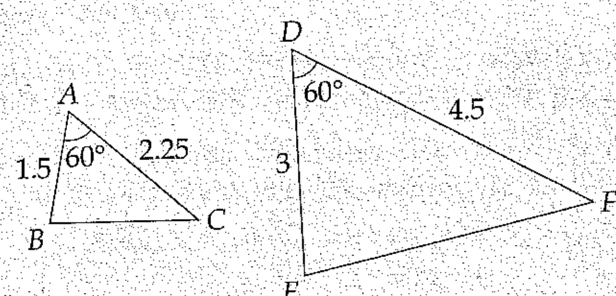
5.3 SAS Similarity

Problems

Problem 5.8:

(a) Measure $\overline{BC}$, $\overline{EF}$, and angles $\angle B$, $\angle C$, $\angle E$, and $\angle F$.

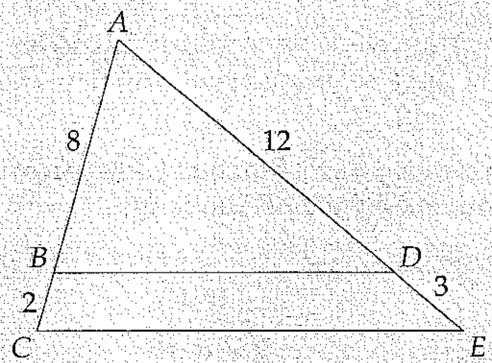
(b) Can you make a guess about how to use Side-Angle-Side for triangle similarity?

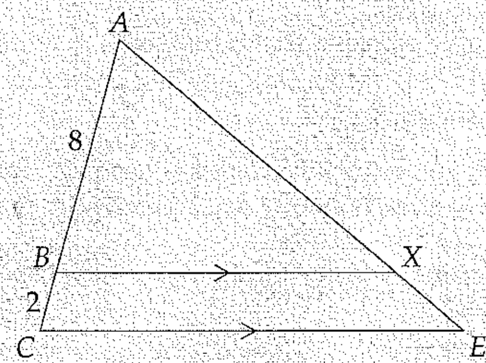


Extra! *Descartes commanded the future from his study more than Napoleon from the throne.*

–Oliver Wendell Holmes

Problem 5.9: In the figure below on the left, we have $\frac{AB}{AC} = \frac{AD}{AE} = \frac{4}{5}$, and clearly $\angle BAD = \angle CAE$. We wish to prove that $\triangle ABD \sim \triangle ACE$. (Note that we cannot assume that $\overline{BD} \parallel \overline{CE}$! We have to prove it.)





(a) Suppose we draw a line through B parallel to $\overline{CE}$ that hits $\overline{AE}$ at X as in the diagram on the right. What do we know about $\triangle ABX$ and $\triangle ACE$?

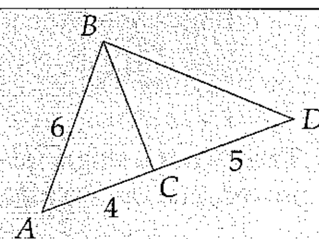
(b) Given that $AE = 15$ in both diagrams above, what is AX?

(c) What can we conclude about D and X?

(d) What can we conclude about $\triangle ABD$ and $\triangle ACE$?

(e) What similarity rule can we create from this investigation?

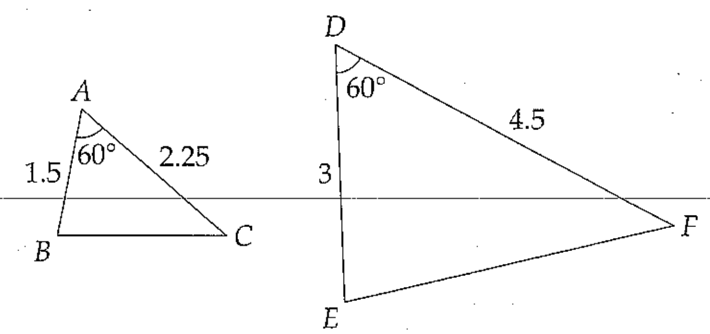
Problem 5.10: Given $AC = 4$, $CD = 5$, and $AB = 6$ as in the diagram, find BC if the perimeter of $\triangle BCD$ is 20. *(Source: Mandelbrot)*



Problem 5.8:

(a) Measure $\overline{BC}$, $\overline{EF}$, and angles $\angle B$, $\angle C$, $\angle E$, and $\angle F$.

(b) Can you make a guess about how to use Side-Angle-Side for triangle similarity?



Solution for Problem 5.8: We aren't surprised to find that BC appears to be half EF: BC is about 2 cm and EF is around 4 cm. We also aren't shocked to find that $\angle B$ appears to equal $\angle E$ and $\angle C$ appears to equal $\angle F$.

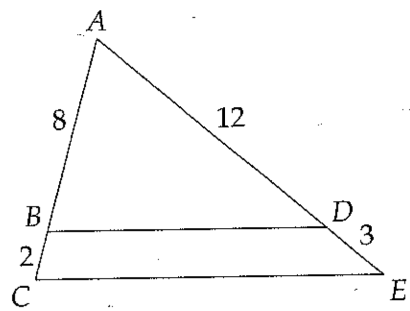
This example suggests that if two sides in one triangle are in the same ratio as two sides in another triangle (as $AB/AC = DE/DF$), and the angles between these sides are equal (as $\angle A = \angle D$), then the triangles are similar. □

No doubt, you know where this is headed. Time to develop a proof for our guess. As usual, we try to use what we already know, AA Similarity, to prove our guess for 'SAS Similarity'.

Problem 5.9: In the figure on the right, we have

$$\frac{AB}{AC} = \frac{AD}{AE} = \frac{4}{5},$$

and clearly $\angle BAD = \angle CAE$. Prove that $\triangle ABD \sim \triangle ACE$.



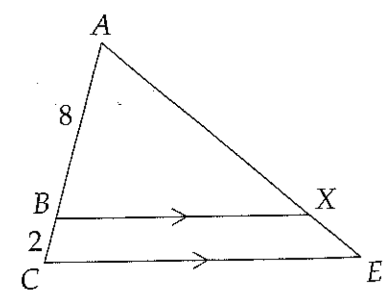
Solution for Problem 5.9: What did we do wrong here:

Bogus Solution: Since $\overline{BD} \parallel \overline{CE}$, we have $\angle ABD = \angle ACE$ and $\angle ADB = \angle AEC$, so $\triangle ABD \sim \triangle ACE$ by AA Similarity.

There's not a single false statement in that solution. However, the assertion that $\overline{BD} \parallel \overline{CE}$ needs to be proved, and our Bogus Solution merely states it without justification.

In the solution below, we take the clever tactic of considering the point X on $\overline{AE}$ such that $\overline{BX} \parallel \overline{CE}$. Then we prove that X is in fact D.

We'd like to prove that $\overline{BD} \parallel \overline{CE}$, but there's no obvious way to even start. We seem stuck, so we try to go a different direction. We create a point X on $\overline{AE}$ as shown at right, such that $\overline{BX} \parallel \overline{CE}$. Our goal now is to show that X must be D. Notice that we are not assuming that $\overline{BD} \parallel \overline{CE}$. We are taking some other point, X, such that $\overline{BX} \parallel \overline{CE}$, then trying to prove that X *must be* D.



Since $\overline{BX} \parallel \overline{CE}$, we have $\angle ABX = \angle ACE$ and $\angle AXB = \angle AEC$, so $\triangle ABX \sim \triangle ACE$ by AA Similarity. Therefore,

$$\frac{AX}{AE} = \frac{AB}{AC} = \frac{4}{5},$$

so $AX = (4/5)(AE) = 12$. Hence, X is on $\overline{AE}$ 12 units from A. But that's where point D is! Therefore, D must be the same point as point X; i.e., D is the point on $\overline{AE}$ such that $\overline{BD} \parallel \overline{CE}$. Now that we've proved $\overline{BD} \parallel \overline{CE}$, we can conclude that $\triangle ABD \sim \triangle ACE$. □

We have established another way to prove two triangles are similar.

> **Important:** **Side-Angle-Side Similarity (SAS Similarity)** tells us that if two sides in one triangle are in the same ratio as two sides in another triangle (as $AB/AC = DE/DF$ below), and the angles between these sides are equal (as $\angle A = \angle D$ below), then the triangles are similar.
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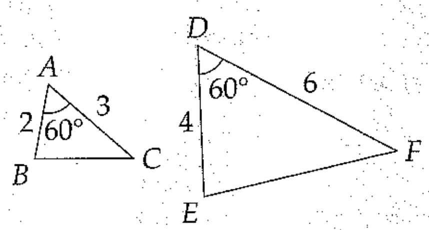
> 
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>
> Note that we can also write that ratio equality as the ratio of corresponding sides in the triangles: $AB/DE = AC/DF$.

You may be wondering how our solution to Problem 5.9 can be used to prove SAS Similarity in general, since Problem 5.9 only deals with the case of two triangles that share an angle, as $\triangle ABD$ and $\triangle ACE$ share $\angle A$. We can use this approach generally because if an angle in one triangle equals an angle in another, we can always slide (and/or flip) one triangle until it's on top of the other, as shown in Figure 5.2.

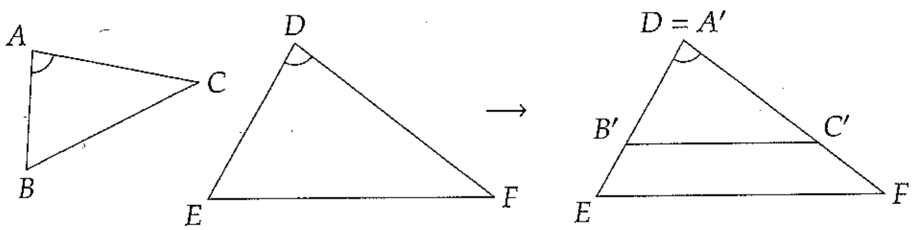
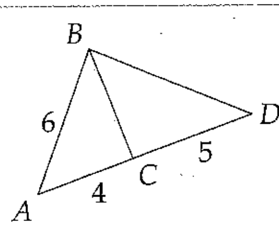


Figure 5.2: Sliding Triangles to Prove Similarity

SAS Similarity is most often used in diagrams like the one shown in Problem 5.9. However, it does come up in less obvious situations.

Problem 5.10: Given $AC = 4$, $CD = 5$, and $AB = 6$ as in the diagram, find BC if the perimeter of $\triangle BCD$ is 20. *(Source: Mandelbrot)*



Solution for Problem 5.10: Since

$$\frac{AC}{AB} = \frac{4}{6} = \frac{2}{3} \quad \text{and} \quad \frac{AB}{AD} = \frac{6}{9} = \frac{2}{3},$$

we have $\triangle ACB \sim \triangle ABD$ by SAS (since the angle between the sides in each ratio above is $\angle A$). Since the sides of $\triangle ABD$ are $3/2$ the corresponding sides of $\triangle ACB$, we have $BD = 3BC/2$. Now we can use that perimeter information. Since $BC + CD + DB = 20$, we have

$$BC + 5 + \frac{3BC}{2} = 20.$$

Therefore, $BC = 6$. □

Exercises

5.3.1 Find DE in the figure at left below.

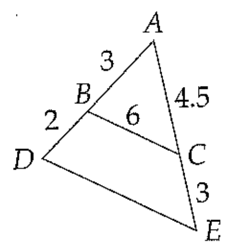


Figure 5.3: Diagram for Problem 5.3.1

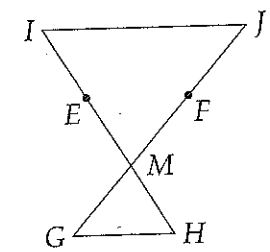


Figure 5.4: Diagram for Problem 5.3.2

5.3.2 In the figure at right above, M is the midpoint of $\overline{EH}$ and of $\overline{FG}$. E and F are midpoints of $\overline{IM}$ and $\overline{MJ}$, respectively. Prove that $\overline{IJ} \parallel \overline{GH}$.

5.3.3 Show that if $WZ^2 = (WX)(WY)$ in the diagram at left below, then $\angle WZX = \angle WYZ$. **Hints:** 147

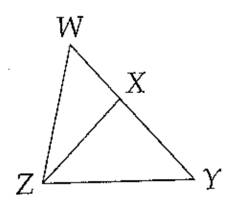


Figure 5.5: Diagram for Problem 5.3.3

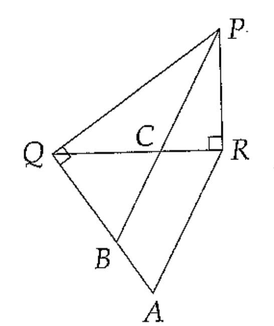


Figure 5.6: Diagram for Problem 5.3.4

5.3.4 In the diagram at right above, $\angle PRQ = \angle PQA = 90°$, $QR = QA$, and $\angle QPC = \angle RPC$.

(a) Prove $\angle QCB = \angle QBC$. **Hints:** 202

(b)★ Prove $\overline{RA} \parallel \overline{PB}$. **Hints:** 388

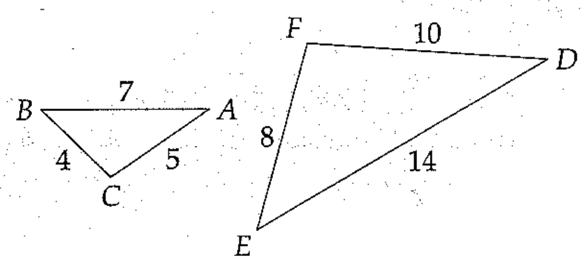
Extra! *I must study politics and war that my children may have liberty to study mathematics and philosophy. My children ought to study mathematics and philosophy, geography, natural history, naval architecture, navigation, commerce, and agriculture, in order to give their children a right to study painting, poetry, music, architecture, statuary, tapestry, and porcelain.*

–John Adams

5.4 SSS Similarity

We use SSS Similarity less often than AA and SAS.

Important: **Side-Side-Side Similarity (SSS Similarity)** tells us that if each side of one triangle is the same constant multiple of the corresponding side of another triangle, then the triangles are similar. (And therefore, their corresponding angles are equal.)



For example, in the diagram, we have

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF},$$

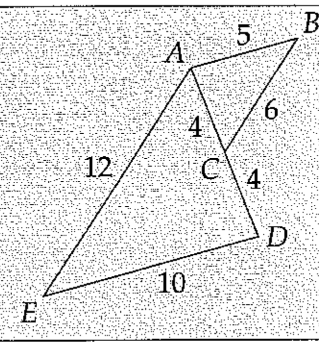
so

$$\triangle ABC \sim \triangle DEF.$$

Therefore, $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.

Problems

Problem 5.11: Given the side lengths shown in the diagram, prove that $\overline{AE} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DE}$.

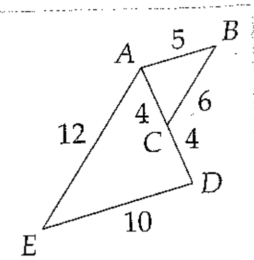


As we noted, few problems require SSS Similarity. We may, however, consider it in problems in which all we are given is lengths, but we have to prove something about angles.

Extra! *Mathematics is the art of giving the same name to different things.*

–Henri Poincaré

Problem 5.11: Given the side lengths shown in the diagram, prove that $\overline{AE} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DE}$.



Solution for Problem 5.11: We need to use angles to show the segments are parallel, but all we have are sides. We look for similarity, and see that

$$\frac{AB}{DE} = \frac{AC}{AD} = \frac{BC}{AE} = \frac{1}{2},$$

so $\triangle ABC \sim \triangle DEA$ by SSS Similarity. Therefore, $\angle BAC = \angle EDA$, so $\overline{AB} \parallel \overline{DE}$. Also, $\angle DAE = \angle ACB$, so $\overline{AE} \parallel \overline{BC}$. □

Exercises

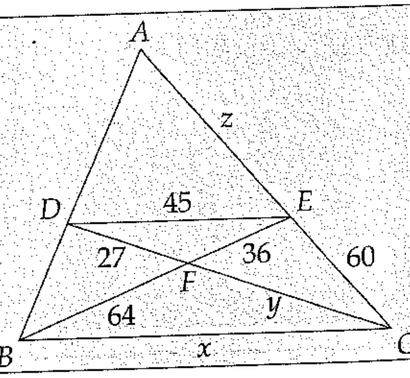
5.4.1 Two isosceles triangles have the same ratio of leg length to base length. Prove that the vertex angles of the two triangles are equal. **Hints:** 314

5.5 Using Similarity in Problems

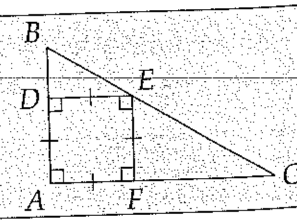
In this section we explore some challenging problems that are solved with similar triangles, and we discover why AA Similarity works.

Problems

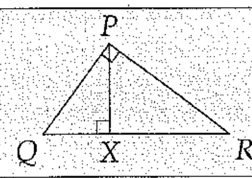
Problem 5.12: In the diagram, $\overline{DE} \parallel \overline{BC}$, and the segments have the lengths shown in the diagram. Find x, y, and z.



Problem 5.13: As shown in the diagram, $\angle A = 90^\circ$ and $ADEF$ is a square. Given that $AB = 6$ and $AC = 10$, find AD.



Problem 5.14: In the diagram, $\overline{PX}$ is the altitude from right angle $\angle QPR$ of right triangle PQR as shown. Show that $PX^2 = (QX)(RX)$, $PR^2 = (RX)(RQ)$, and $PQ^2 = (QX)(QR)$.



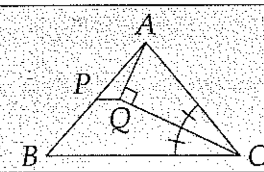
Problem 5.15: $\triangle ABC \sim \triangle XYZ$, $AB/XY = 4$, and $[ABC] = 64$. In this problem we will find $[XYZ]$.

(a) Let h_C be the altitude of $\triangle ABC$ to AB, and let h_Z be the altitude of $\triangle XYZ$ to XY. What is h_C/h_Z?

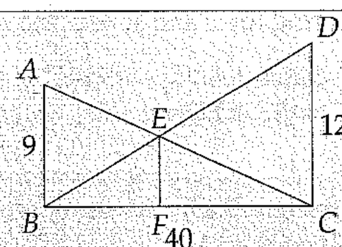
(b) Find $[XYZ]$.

Extra challenge: What general statement about the areas of similar triangles can you make?

Problem 5.16: In the diagram, $\angle ACQ = \angle QCB$, $\overline{AQ} \perp \overline{CQ}$, and P is the midpoint of $\overline{AB}$. Prove that $\overline{PQ} \parallel \overline{BC}$. **Hints:** 35, 417



Problem 5.17: Flagpole $\overline{CD}$ is 12 feet tall. Flagpole $\overline{AB}$ is 9 feet tall. Both flagpoles are perpendicular to the ground. A straight wire is attached from B to D, and another from A to C. The flagpoles are 40 feet apart, and the wires cross at E, which is directly above point F on the ground. We wish to find EF.



(a) Use similar triangles to find ratios of segments that equal EF/AB.

(b) Use similar triangles to find ratios of segments that equal EF/DC.

(c) Cleverly choose one ratio from each of the first two parts and add them to get an equation you can solve for EF.

Problem 5.18: In this problem we will explore why AA Similarity works. Do not use AA Similarity to solve the problem!

In the diagram below, we have two triangles ($\triangle ABE$ and $\triangle ACD$) with equal angles, and sides with lengths as marked. Our goal in this problem is to find BE and DE, and discover a process to prove that if the angles of one triangle equal those of another, then the corresponding sides of the two triangles are in constant proportion. We will make heavy use of the Same Base/Same Altitude principle we discovered in Section 4.3, so you might want to review that section if you get stuck.

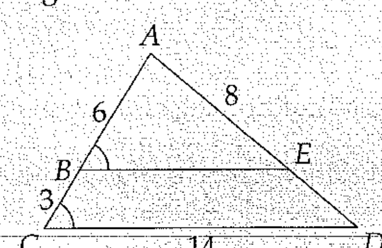
(a) What are $[ABE]/[ACE]$ and $[BEC]/[BED]$?

(b) Use the previous part to show that $[ACE] = [ABD]$.

(c) What is $[ABE]/[ABD]$?

(d) Use the previous part to find AD.

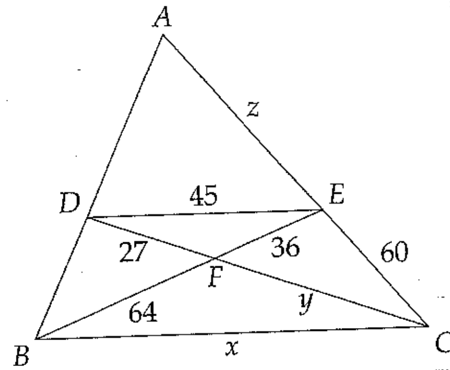
(e) What is BE?



(f) Can we use our work in this problem to prove that if two angles of one triangle equal those of another triangle, then the triangles are similar?

We start off with some warm-ups involving parallel and perpendicular lines.

Problem 5.12: In the diagram, $\overline{DE} \parallel \overline{BC}$, and the segments have the lengths shown in the diagram. Find x, y, and z.



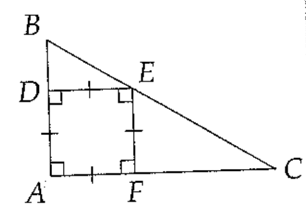
Solution for Problem 5.12: Since $\overline{ED} \parallel \overline{BC}$, we have $\triangle FBC \sim \triangle FED$ by AA Similarity. Therefore, we have

$$\frac{FC}{FD} = \frac{BC}{DE} = \frac{FB}{FE} = \frac{64}{36} = \frac{16}{9}.$$

Solving for x and y, we find $x = BC = (16/9)(DE) = 80$ and $y = FC = (16/9)(DF) = 48$.

Since $\triangle ADE \sim \triangle ABC$ by AA Similarity, we have $AE/AC = DE/BC = 45/80 = 9/16$. Since $AE = z$ and $AC = AE + EC = z + 60$, we have $z/(z+60) = 9/16$. Cross-multiplying gives $16z = 9z + 540$, so $z = 540/7$. □

Problem 5.13: As shown in the diagram, $\angle A = 90°$, and $ADEF$ is a square. Given that $AB = 6$ and $AC = 10$, find AD.



Solution for Problem 5.13: Since $\angle A = \angle EFC = 90°$, we have $\overline{EF} \parallel \overline{AB}$; similarly, $\overline{DE} \parallel \overline{AC}$. Therefore, this problem has both right triangles and parallel lines. Our parallel lines quickly tell us that by AA, we have

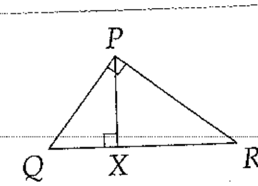
$$\triangle BDE \sim \triangle BAC \sim \triangle EFC.$$

If we let each side of $ADEF$ be x, we have $BD = 6 - x$ and $FC = 10 - x$. Our similar triangles can then be used to solve for x. From $\triangle BDE \sim \triangle EFC$, we have $BD/DE = EF/FC$. Substitution gives

$$\frac{6-x}{x} = \frac{x}{10-x}.$$

Cross-multiplying and solving the resulting equation for x gives $x = 15/4$. Therefore, $AD = x = 15/4$. □

Problem 5.14: In the diagram, $\overline{PX}$ is the altitude from right angle $\angle QPR$ of right triangle PQR as shown. Show that $PX^2 = (QX)(RX)$, $PR^2 = (RX)(RQ)$, and $PQ^2 = (QX)(QR)$.



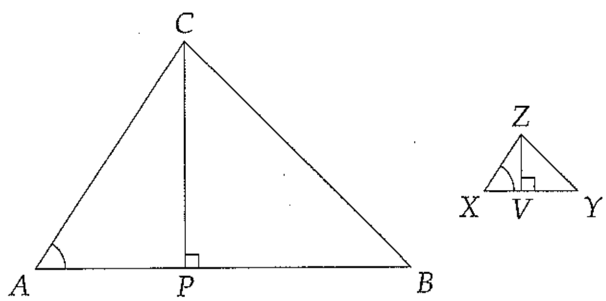
Solution for Problem 5.14: Right triangles mean similar triangles. $\angle PXR = \angle QPR$ and $\angle PRX = \angle PRQ$, so $\triangle PXR \sim \triangle QPR$. Therefore, we have $PR/RX = RQ/PR$, so $PR^2 = (RX)(RQ)$. Similarly, we can show $\triangle PQX \sim \triangle RQP$, so $PQ/QX = QR/PQ$, and we have $PQ^2 = (QX)(QR)$.

Combining the two triangle similarities (or by noting that $\angle XPQ = 90^\circ - \angle XPR = \angle XRP$ and $\angle PXQ = \angle PXR$), we find $\triangle PXQ \sim \triangle RXP$. Therefore, $PX/QX = RX/PX$, so $PX^2 = (RX)(QX)$. □

The square root of the product of two numbers is called the **geometric mean** of the two numbers. The previous problem suggests where the name 'geometric mean' comes from. For example, what is the geometric mean of QX and RX?

Problem 5.15: Given that $\triangle ABC \sim \triangle XYZ$, $AB/XY = 4$, and $[ABC] = 64$, find $[XYZ]$.

Solution for Problem 5.15: Since $\triangle ABC \sim \triangle XYZ$ and $AB/XY = 4$, the ratio of corresponding lengths in the triangles is $4/1$. Therefore, the altitude of $\triangle ABC$ to $\overline{AB}$ is 4 times the corresponding altitude to $\overline{XY}$ in $\triangle XYZ$.



For a quick proof, consider the diagram to the right, in which we've drawn the aforementioned altitudes to $\overline{AB}$ and $\overline{XY}$. Since $\triangle ABC \sim \triangle XYZ$, we have $\angle A = \angle X$. Combining this angle equality with $\angle CPA = \angle ZVX$ gives $\triangle APC \sim \triangle XVZ$ by AA, so $CP/ZV = AC/XZ$. Since $\overline{AC}$ and $\overline{XZ}$ are corresponding sides of our original triangles, their ratio is $4/1$, so $CP/ZV = 4/1$.

Finally, we can find the ratio $[ABC]/[XYZ]$. Since both the base and the altitude of $\triangle ABC$ are 4 times the corresponding base and altitude of $\triangle XYZ$, we know that

$$[ABC]/[XYZ] = \frac{(AB)(CP)/2}{(XY)(ZV)/2} = \left(\frac{AB}{XY}\right)\left(\frac{CP}{ZV}\right) = \left(\frac{4}{1}\right)^2 = 16.$$

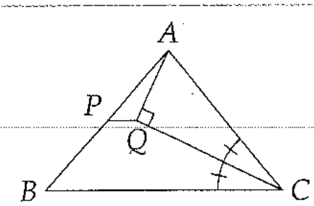
So, we have $[XYZ] = [ABC]/16 = 4$. □

The same procedure we used to solve this problem can be used to find an important relationship between the areas of two similar triangles.

Important: If two triangles are similar such that the sides of the larger triangle are k times the sides of the smaller, then the area of the larger triangle is k^2 times that of the smaller.

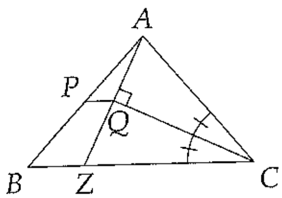
This relationship holds for any pair of similar figures, not just for triangles.

Problem 5.16: In the diagram, $\angle ACQ = \angle QCB$, $\overline{AQ} \perp \overline{CQ}$, and P is the midpoint of $\overline{AB}$. Prove that $\overline{PQ} \parallel \overline{BC}$.



Solution for Problem 5.16: If we could show that $\angle QPA = \angle B$, then we could use that to prove $\overline{PQ} \parallel \overline{BC}$.

Unfortunately, there are no obvious similar triangles or congruent triangles we can use to show that $\angle QPA = \angle B$.



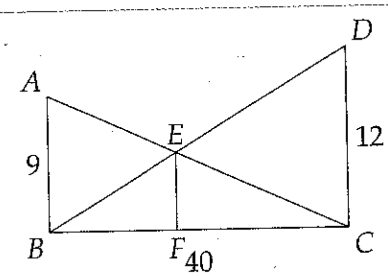
We extend segment $\overline{AQ}$ to point Z on $\overline{BC}$ because we'd like to create triangles that might be similar (namely, $\triangle APQ$ and $\triangle ABZ$). We'd also like to use the angle equalities at C, which we can now do by noting that $\angle AQC = \angle CQZ$, $CQ = CQ$, and $\angle ACQ = \angle QCZ$, so $\triangle CQZ \cong \triangle CQA$ by ASA. Therefore, we know that $AQ = QZ$, so $AQ = AZ/2$.

We might seem stuck here, but then we remember the last bit of information we haven't used. Since P is the midpoint of $\overline{AB}$, we have $AP = AB/2$, so $\triangle PAQ \sim \triangle BAZ$ by SAS Similarity. Thus, $\angle APQ = \angle B$, so $\overline{PQ} \parallel \overline{BC}$. □

Concept: When you're stuck on a problem, ask yourself, 'What piece of information have I not used?'

Concept: In many problems, there's more than meets the eye. Extending segments that seem to end abruptly (particularly in the middle of a triangle) sometimes gives useful information.

Problem 5.17: Flagpole $\overline{CD}$ is 12 feet tall. Flagpole $\overline{AB}$ is 9 feet tall. Both flagpoles are perpendicular to the ground. A straight wire is attached from B to D, and another from A to C. The flagpoles are 40 feet apart, and the wires cross at E, which is directly above point F on the ground. Find EF.



Solution for Problem 5.17: We start off by noticing that $\overline{AB} \parallel \overline{EF} \parallel \overline{CD}$ since all three are perpendicular to $\overline{BC}$. By now you know the drill: parallel lines mean similar triangles. We look first for similar triangles that include $\overline{EF}$, and we see $\triangle CEF \sim \triangle CAB$ and $\triangle EBF \sim \triangle DBC$. Therefore, we have

$$\frac{EF}{AB} = \frac{CF}{CB} = \frac{EC}{AC} \quad \text{and} \quad \frac{EF}{CD} = \frac{BF}{CB} = \frac{BE}{BD}.$$

We see CB in both groups, so we investigate the ratios involving CB more closely. We see that we have $CF + BF = CB$, so

$$\frac{EF}{AB} + \frac{EF}{CD} = \frac{CF}{CB} + \frac{BF}{CB} = \frac{CF + BF}{CB} = \frac{CB}{CB} = 1.$$

Now we can find EF:

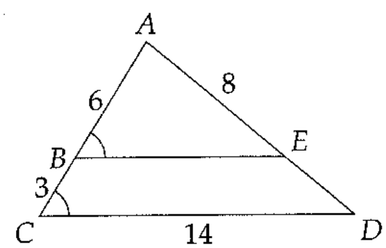
$$EF = \frac{1}{\frac{1}{AB} + \frac{1}{CD}} = \frac{1}{\frac{1}{9} + \frac{1}{12}} = \frac{36}{7}.$$

Notice that the length of $\overline{BC}$ is irrelevant! □

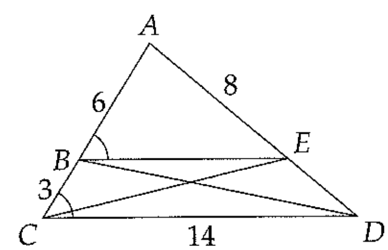
WARNING!! In the last solution we didn't spell out exactly why $\triangle EBF \sim \triangle DBC$, since we've gone through those steps several times already. When you are writing solutions for your class or for a contest, you should include the steps we left out here (cite which angles are equal and why, then invoke AA). Only start leaving out the simple steps if you are certain that it is O.K. to do so.

We finish this section by exploring why AA Similarity works.

Problem 5.18: In the diagram we have two triangles ($\triangle ABE$ and $\triangle ACD$) with equal angles, and sides with lengths as marked. Find BE and DE without using AA Similarity. Can you use your method to prove why AA Similarity works?



Solution for Problem 5.18: We would like to show that $\triangle ABE \sim \triangle ACD$, so we start thinking about side length ratios. The only ratio tool we have that doesn't depend on having similar triangles already is our Same Base/Same Altitude technique of Section 4.3, so we try that. We don't have any triangles to use our technique on, so we draw $\overline{BD}$ and $\overline{CE}$ as shown to the right.



$\triangle ABE$ and $\triangle AEC$ share an altitude from E, so

$$\frac{[ABE]}{[ACE]} = \frac{AB}{AC} = \frac{6}{9} = \frac{2}{3}. \tag{5.1}$$

Similarly, $\triangle ABE$ and $\triangle ABD$ share an altitude from B, so

$$\frac{[ABE]}{[ABD]} = \frac{AE}{AD} = \frac{8}{8+DE}. \tag{5.2}$$

We suspect that $AB/AC = AE/AD$ because we suspect $\triangle ABE \sim \triangle ACD$. From (5.1) and (5.2), we have

$$\frac{AE}{AD} = \frac{[ABE]}{[ABD]} \quad \text{and} \quad \frac{AB}{AC} = \frac{[ABE]}{[ACE]}.$$

Since the numerators in our area ratios are the same, we need only show that $[ABD] = [ACE]$. These two areas share $[ABE]$, so we need only show that $[BEC] = [BED]$.

Since $\angle ABE = \angle ACD$, we know $\overline{BE} \parallel \overline{CD}$. Therefore, the altitudes from C and D to $\overleftrightarrow{BE}$ must be the same. Hence, triangles BEC and BED have the same base ($\overline{BE}$) and the same length altitudes to that base, so $[BEC] = [BED]$.

Finally, we can find DE. We have:

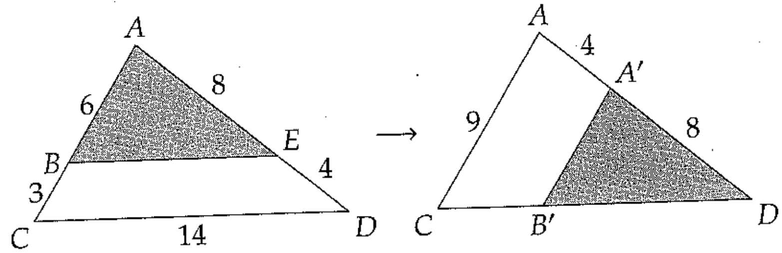
$$[AEC] = [AEB] + [BEC] = [AEB] + [BED] = [ABD],$$

so we can use our area ratios above. Since $[ABE]/[ACE] = [ABE]/[ABD]$, we have $AB/AC = AE/AD$, so

$$2/3 = 8/(8 + DE).$$

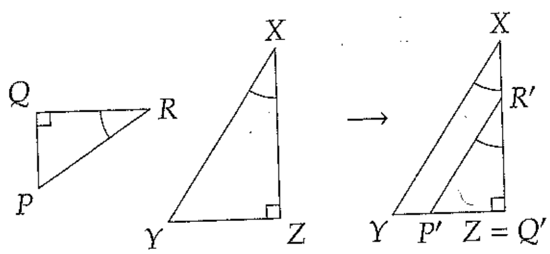
Solving this equation for DE, we find that $DE = 4$.

Now we very strongly suspect that $BE/CD = AE/AD$. To prove it, we use the same process we just followed.



We consider $\triangle A'B'D$ where A' is on $\overline{AD}$ and B' is on $\overline{CD}$ such that $A'D = AE$ and $BE = B'D$. Since $\overline{BE} \parallel \overline{CD}$, we have $\angle AEB = \angle ADC = \angle A'DB'$. Therefore, $\triangle A'B'D \cong \triangle ABE$ by SAS Congruence. (You can also think of $\triangle A'DB'$ as the result of sliding $\triangle ABE$ along $\overline{AD}$ until side $\overline{BE}$ is on $\overline{CD}$.) Since $A'D = AE$, we have $AA' = ED = 4$. We also have $\angle DA'B' = \angle EAB = \angle DAC$, so $\overline{AC} \parallel \overline{A'B'}$.

We can chase areas around as before to show that $B'D/CD = A'D/AD$, so $B'D = (2/3)(14) = 28/3$. Since $B'D = BE$, we have $BE = 28/3$. Note that because $B'D = BE$ and $A'D = AE$, we have shown that $BE/CD = AE/AD$, as suspected. □



Whenever we have a two triangles that have two angle measures in common, we can slide (and possibly flip) one triangle onto the other so that we get a diagram like that in Problem 5.18. For example, $\triangle PQR$ and $\triangle YZX$ in the diagram to the left have two angle measures in common (and consequently the third angles are equal, too). We can therefore move $\triangle PQR$ on top of $\triangle YZX$ such that two of the sides of the 'moved' triangle coincide with sides of $\triangle YZX$, as $\triangle P'Q'R'$ in the diagram shows.

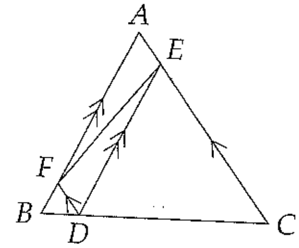
We can use the exact same approach as we used in Problem 5.18 to show that if two angles of one triangle equal the corresponding angles of the other, then each pair of corresponding lengths in the two triangles has the same ratio.

Concept: Area can be a very useful problem solving tool even in problems that appear to have nothing to do with area.

Exercises

5.5.1 X and Y are on sides $\overline{PQ}$ and $\overline{PR}$, respectively, of $\triangle PQR$ such that $\overline{XY} \parallel \overline{QR}$. Given $XY = 5$, $QR = 15$, and $YR = 8$, find PY.

5.5.2 In the figure, the area of $\triangle EDC$ is 25 times the area of $\triangle BFD$.

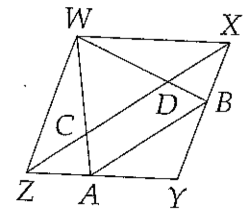


(a) Find CD/DB. **Hints:** 350

(b) Find $[EDC]/[ABC]$. **Hints:** 171

(c)★ Find $[AFE]/[ABC]$. **Hints:** 321

5.5.3 In the diagram, $\overline{WZ} \parallel \overline{XY}$ and $\overline{WX} \parallel \overline{ZY}$. $\overline{WA}$ and $\overline{WB}$ hit $\overline{XZ}$ at C and D, respectively, such that $ZC = XD$.



(a) Prove that $ZC/XC = AC/WC$.

(b) Prove that $XD/ZD = DB/WD$.

(c) Prove $\overline{CD} \parallel \overline{AB}$. **Hints:** 462

5.5.4 In the diagram at left below, $PQ = PR$, $\overline{ZX} \parallel \overline{QY}$, $\overline{QY} \perp \overline{PR}$, and $\overline{PQ}$ is extended to W such that $\overline{WZ} \perp \overline{PW}$.

(a) Show that $\triangle QWZ \sim \triangle RXZ$. **Hints:** 360

(b)★ Show that $YQ = ZX - ZW$. **Hints:** 172, 550

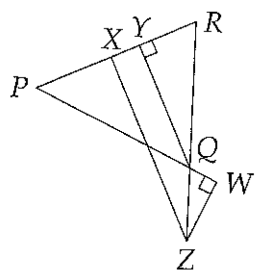


Figure 5.7: Diagram for Problem 5.5.4

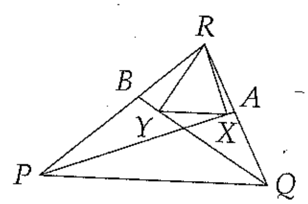


Figure 5.8: Diagram for Problem 5.5.5

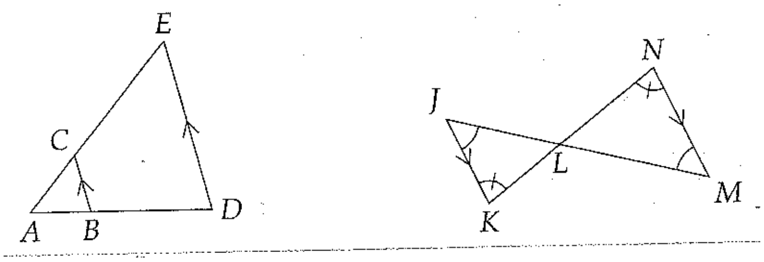
5.5.5★ $\overline{PA}$ and $\overline{BQ}$ bisect angles $\angle RPQ$ and $\angle RQP$, respectively. Given that $\overline{RX} \perp \overline{PA}$ and $\overline{RY} \perp \overline{BQ}$, prove that $\overline{XY} \parallel \overline{PQ}$. **Hints:** 584, 152, 254

Concepts: . . . *continued from the previous page*

- In many problems, there's more than meets the eye. Extending segments that seem to end abruptly (particularly in the middle of a triangle) can often yield quick solutions.
- When stuck on a problem, try solving an easier related problem. For constructions, useful easier related problems often involve relaxing one of the constraints of the problem.
- Consider using similar triangles in problems involving ratios of segment lengths.

Things To Watch Out For!

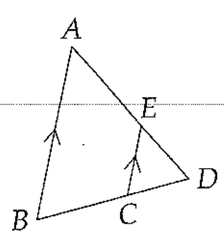
WARNING!! Below are shown two common situations that lead to mistakes. The diagram on the left may lead you to write '$\triangle ABC \sim \triangle ADE$, so $AB/BD = BC/DE$.' The one on the right might lead to '$\triangle JKL \sim \triangle NLM$,' so $JL/NL = KL/ML$.' Both of these are **incorrect**! Make sure you see why!



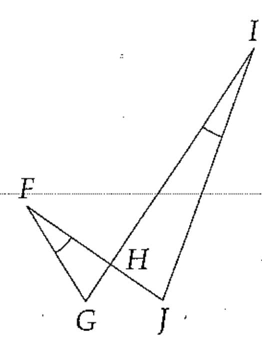
REVIEW PROBLEMS

5.22 In each of the parts below, either identify all pairs of similar triangles or state that there are not any pairs of triangles that are necessarily similar. For each pair of similar triangles you find, state why the triangles are similar.

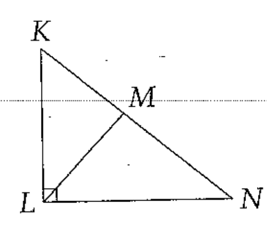
(a)

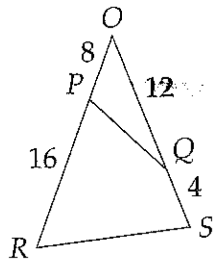


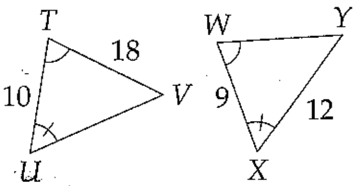
(b)

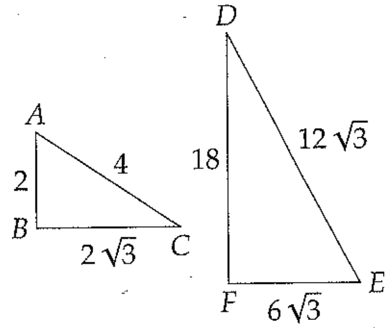


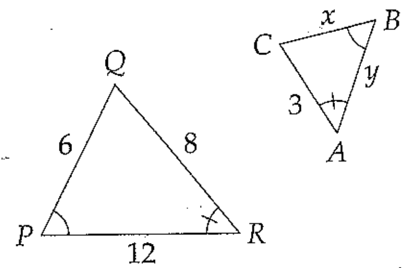
(c)



(d)


(e)


(f)




.23 Find x and y in the diagram at right, given the angle equalities nd side lengths shown in $\triangle PQR$ and $\triangle ABC$.

.24 Points P and Q are on $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{PQ} \parallel \overline{BC}$. iven $AB = 12$, $PB = 9$, and $AC = 18$, find QA.

.25 The side lengths of a triangle are 4 centimeters, 6 centimeters, nd 9 centimeters. One of the side lengths of a similar triangle is 36 entimeters. What is the maximum number of centimeters possible in the perimeter of the second iangle? *(Source: MATHCOUNTS)*

.26 What's wrong with the diagram shown at left below?

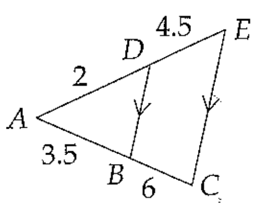


Figure 5.9: Diagram for Problem 5.26

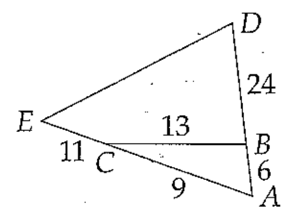


Figure 5.10: Diagram for Problem 5.27

.27 Find DE in the diagram at right above.

.28 Why is the diagram shown at left below impossible?

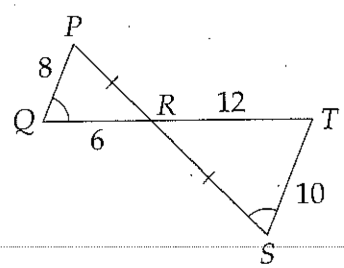


Figure 5.11: Diagram for Problem 5.28

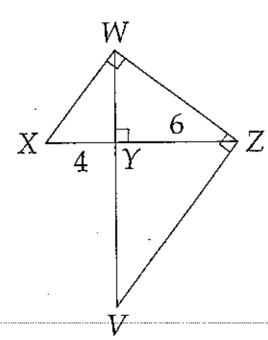
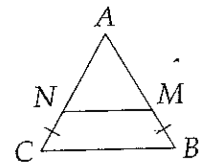


Figure 5.12: Diagram for Problem 5.29

.29 In the diagram at right above, find WY and YV.



5.30 $\triangle ABC$ at right is equilateral. M is on $\overline{AB}$ and N on $\overline{AC}$ such that $BM = CN$.

(a) Prove that $AM = AN$.

(b) Prove that $\triangle AMN$ is equilateral.

5.31 Given $\triangle ABC \sim \triangle YZX$, $[ABC] = 40$, $[YZX] = 360$, $AB = 9$, and $BC = 12$, find the following:

(a) YZ.

(b) The length of the altitude to side $\overline{XZ}$ of triangle $\triangle YZX$.

5.32 Let $ABCD$ be a rectangle as shown at left below, with $AB = 25$ and $BC = 12$. Let E be a point on $\overline{AB}$, such that $AE < BE$ and triangles AED and BCE are similar. Find AE.

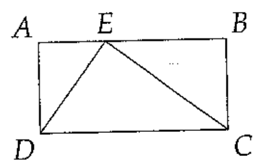


Figure 5.13: Diagram for Problem 5.32

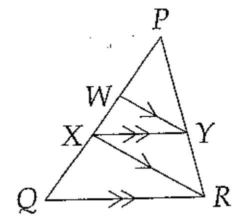


Figure 5.14: Diagram for Problem 5.33

5.33 In the diagram at right above, $PW = 6$ and $WX = 4$. Find QX.

5.34 *(Try this without looking back in the text first!)* In the diagram at left below, $\overline{AP} \parallel \overline{BQ} \parallel \overline{CR}$. Prove that

$$\frac{1}{CR} = \frac{1}{AP} + \frac{1}{BQ}.$$

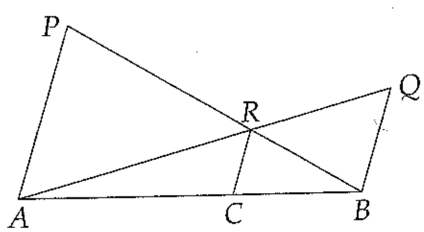


Figure 5.15: Diagram for Problem 5.34

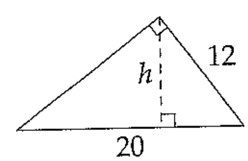


Figure 5.16: Diagram for Problem 5.35

5.35 Two of the sides in the right triangle at right above have length 12 cm and 20 cm, as shown. What is the number of centimeters in the length of the altitude h drawn to the side with length 20 cm? *(Source: MATHCOUNTS)* *(If you know the Pythagorean Theorem, try doing this problem without it!)*

5.36 Let ABC be a triangle, and let D and E be points on sides $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{DE} \parallel \overline{BC}$. Prove that

$$\frac{AD}{AE} = \frac{DB}{CE}.$$

(Try to do this one without looking back in the text for the proof!)

Challenge Problems

5.37 Let ABC be a triangle, and let D and E be points on $\overline{AB}$ and $\overline{AC}$, respectively, such that $AD/AE = BD/EC$. Prove that $\overline{DE} \parallel \overline{BC}$. *Make sure you see why this differs from the previous problem!* **Hints:** 363, 179

5.38 If the sum of one of the base angles and the vertex angle is the same for two different isosceles triangles, must the triangles be similar? **Hints:** 196

5.39 In the figure at left below, isosceles $\triangle ABC$ with base $\overline{AB}$ has altitude $CH = 24$ cm. $DE = GF$, $HF = 12$ cm, and $FB = 6$ cm. Find the area of $CDEFG$. *(Source: MATHCOUNTS)* **Hints:** 490

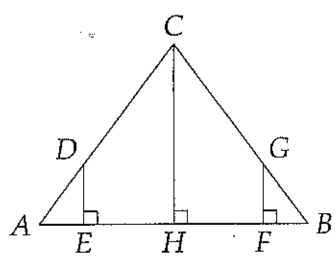


Figure 5.17: Diagram for Problem 5.39

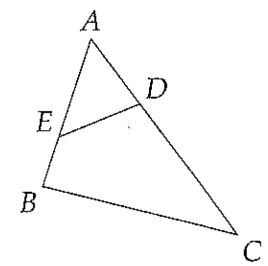


Figure 5.18: Diagram for Problem 5.40

5.40 In triangle ABC at right above, D and E are points on sides $\overline{AC}$ and $\overline{AB}$, respectively, such that $(AD)(AC) = (AE)(AB)$. Prove that $\angle CDE + \angle CBE = 180^\circ$ and $\angle ADB + \angle BEC = 180^\circ$. **Hints:** 269, 70

5.41 D, E, and F are on sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively, of $\triangle ABC$ such that $\overline{DE} \parallel \overline{AB}$, $\overline{DF} \parallel \overline{AC}$, and $\overline{BC} \parallel \overline{EF}$. Prove that D, E, and F are the midpoints of the sides of $\triangle ABC$. **Hints:** 24

5.42 In the diagram at left below, $\overline{PS} \parallel \overline{QT}$ and $\overline{PQ} \parallel \overline{ST}$. Prove that $SU/SP = QP/QR$.

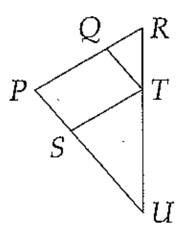


Figure 5.19: Diagram for Problem 5.42

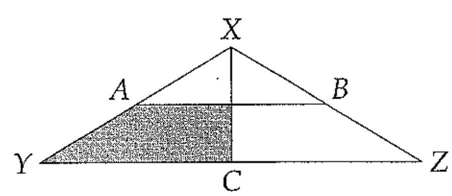


Figure 5.20: Diagram for Problem 5.43

5.43 The area of triangle XYZ at right above is 8 square inches. Points A and B are midpoints of congruent segments $\overline{XY}$ and $\overline{XZ}$. Altitude $\overline{XC}$ bisects $\overline{YZ}$. What is the area of the shaded region? *(Source: AMC 8)* **Hints:** 77, 162

5.44 The midpoints of the three sides of an equilateral triangle are connected to form a second triangle. A third triangle is formed by connecting the midpoints of the second triangle. This process is repeated until a tenth triangle is formed. What is the ratio of the perimeter of the tenth triangle to that perimeter of the third triangle? *(Source: MATHCOUNTS)* **Hints:** 72